

Integrability of the Korteweg-de Vries equation valued on a Cayley-Dickson algebra

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Abstract

We introduce the Korteweg-de Vries equation valued on a Cayley-Dickson algebra. That is, the field whose evolution is determined by KdV equation is valued on a Cayley-Dickson algebra. The product in the equation being the product on the algebra. We show that the resulting evolution equations are integrable in the sense that it possess an infinite sequence of local conserved quantities. The argument follows by considering a Bäcklund transformation in the sense of Walhquist-Estabrook for any Cayley-Dickson algebra and relating it to a generalized Gardner equation. From it, the infinite sequence of conserved quantities follows directly. We give the explicit expression for the first few of them.

Keywords: Integrable systems, conservation laws, partial differential equations, rings and algebras

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1 Introduction

Since the work of Miura and Gardner et al [1, 2, 3, 4, 5], carried out in the context of the Korteweg-de Vries (KdV) equation, a lot of interesting results about the so-called integrable systems have been obtained . The fascinating theory developed from that time involves, among other features, several methods to solve non-linear partial differential

equations which arise in a wide range of mathematical and physical situations. This is the case, for example, of the Hirota bilinear method [6], which allows to obtain multisolitonic solutions and of the Bäcklund transformation obtained by Wahlquist and Estabrook [7] with its corresponding generating formula of solutions (including multisolitonic solutions) which acts as a nonlinear superposition principle.

One important aspect of these systems is the existence of infinite conserved quantities. A nice idea due to Gardner [1] allows to obtain them, for the KdV equation, using simultaneously a parametric auxiliar equation (the Gardner equation) and a parametric transformation (the Gardner transformation), relating solutions between both equations.

In [8] Chen obtained the Wahlquist-Estabrook formulation for the KdV equation starting from the associated Lax equations and using a discrete symmetry of the associated Gardner equation (see also [9]). Satsuma in [10] used the Wahlquist-Estabrook transformation to get the corresponding infinite sequence of conserved quantities.

Following these ideas, it is a natural question to ask which of the possible extensions of the KdV equation still possess the integrability properties of the KdV equation. An integrable Grassmann valued extension of KdV was obtained in [11]. A class of extensions of KdV equation arises by introducing supersymmetry. Several integrable supersymmetric extensions were given in [12, 13, 14, 15, 16, 17]. These extensions are a special subset of the set of the coupled extensions. Coupled extensions of the KdV equation form a category by itself, containing among others the ones giving in [18, 19, 20].

Some extensions of the KdV equation follow by considering the defining field to be valued either in associative algebras [21, 22] or in non-associative algebras arising as non-commutative generalizations of Jordan ones [23]. Another extensions of KdV can be given by considering a supersymmetric extension and then substituting the defining fields by Clifford valued fields [24, 25] (in order to analyze a supersymmetric breaking procedure with nice stability properties for the resulting solitonic solutions) or by operators acting in some general space of functions [26].

In the present work we study the extension corresponding to a field valued on a Cayley-Dickson algebra, containing as a particular case the octonion algebra. The octonion algebra is a non-commutative, non-associative algebra, and alternative normed division algebra.

In connection with physics theories it is important to mention that, in particular, the octonion algebra is directly related to the supersymmetric theories. In particular octonion truncations of the Supermembrane theories are interesting models for describing aspects of the unification of the fundamental forces in nature.

The Cayley-Dickson algebras are constituted by a sequence of algebras, starting with the reals \mathbb{R} , obtained inductively with what is called the Cayley-Dickson process. At every stage of this process, a new algebra with twice the dimension of the previous one is formed by considering pairs of elements in the preceding algebra, with multiplication given by $(p, q)(r, s) = (pr - s^*q, sp + qr^*)$ where $(p, q)^* = (p^*, -q)$ is the conjugation map and $a^* = a$ for $a \in \mathbb{R}$. The first four algebras generated by this process are, precisely,

the normed division algebras: the reals \mathbb{R} , the complex numbers \mathbb{C} , the quaternions \mathbb{H} , and the octonions \mathbb{O} . All further algebras in this process have zero divisors and lack the alternative property; the next algebra in the sequence is known as the sedenions \mathbb{S} . Nonetheless, all of the Cayley-Dickson algebras are power associative.

We will consider in this work an extension of the KdV equation where the field which enters into its definition is valued on an arbitrary Cayley-Dickson algebra and the multiplication in the equation is the multiplication of the corresponding algebra. We will prove that the resulting equation is an integrable equation in the sense that it has an infinite sequence of conserved quantities. To do this we will introduce a Bäcklund transformation in the sense of Wahlquist-Estabrook and relate it to a generalized Gardner equation. From it one can directly obtain the infinite sequence of conserved quantities. In Section 2 we define the KdV on an octonion algebra and analyzed its global symmetries. In Section 3 we use an explicit representation of the algebra of the G_2 exceptional Lie group of automorphisms of the octonions to show the invariance of the KdV equation under this symmetry. In Section 4 we introduce a Bäcklund transformation in the sense of Wahlquist-Estabrook for a KdV equation valued on a Cayley-Dickson algebra. In Section 5 we relate the Wahlquist-Estabrook construction to a generalized Gardner transformation and Gardner equation. In Section 6 we prove the existence of an infinite sequence of conserved quantities. In Section 7 we give the conclusions.

2 Korteweg-de Vries equation valued on the algebra of octonions

We denote $u = u(x, t)$ a function with domain in $\mathbb{R} \times \mathbb{R}$ valued on the octonionic algebra. If we denote $e_i, i = 1, \dots, 7$ the imaginary basis of the octonions, u can be expressed as

$$u(x, t) = b(x, t) + \vec{B}(x, t) \quad (1)$$

where $b(x, t)$ is the real part and $\vec{B} = \sum_{i=1}^7 B_i(x, t)e_i$ its imaginary part.

The KdV equation formulated on the algebra of octonions, or simply the octonion KdV equation, is given by

$$u_t + u_{xxx} + \frac{1}{2}(u^2)_x = 0, \quad (2)$$

when $\vec{B} = \vec{0}$ it reduces to the scalar KdV equation. In terms of b and \vec{B} the equation can be re-expressed as

$$b_t + b_{xxx} + bb_x - \sum_{i=1}^7 B_i B_{ix} = 0, \quad (3)$$

$$(B_i)_t + (B_i)_{xxx} + (bB_i)_x = 0. \quad (4)$$

Equation (2) is invariant under Galileo transformations. In fact, if

$$\begin{aligned}\tilde{x} &= x + ct, \\ \tilde{t} &= t, \\ \tilde{u} &= u + c\end{aligned}$$

where c is a real constant, then $\frac{\partial}{\partial x} = \frac{\partial}{\partial \tilde{x}}$ and $\frac{\partial}{\partial t} = c\frac{\partial}{\partial \tilde{x}} + \frac{\partial}{\partial \tilde{t}}$.

We thus have

$$\begin{aligned}u_t &= c\tilde{u}_{\tilde{x}} + \tilde{u}_{\tilde{t}}, \\ u_{xxx} &= \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}}, \\ u_x u + u u_x &= \tilde{u}_{\tilde{x}} \tilde{u} + \tilde{u} \tilde{u}_{\tilde{x}} - 2c\tilde{u}_{\tilde{x}}.\end{aligned}$$

After replacing into (2) we obtain the same equation for $\tilde{u}(\tilde{x}, \tilde{t})$,

$$\tilde{u}_{\tilde{t}} + \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} + \frac{1}{2}(\tilde{u}^2)_{\tilde{x}} = 0.$$

Additionally, Equation (2) is invariant under the automorphisms of the octonions, that is, under the group G_2 . If under an automorphism

$$u \rightarrow \phi(u)$$

then

$$u_1 u_2 \rightarrow \phi(u_1 u_2) = \phi(u_1) \phi(u_2)$$

and consequently

$$[\phi(u)]_t + [\phi(u)]_{xxx} + \frac{1}{2}([\phi(u)]^2)_x = 0.$$

We explicitly analyze the symmetry under the group G_2 in the next section.

3 The derivation algebra

According to Cartan's classification of simple Lie groups, G_2 is the smallest exceptional Lie group. It is the group of automorphisms of the octonions. The tangent space to a group of automorphisms is an algebra of derivations. Therefore the Lie algebra \mathfrak{g}_2 of the Lie group G_2 is $Der(\mathbb{O})$. The elements in $Der(\mathbb{O})$ can be expressed as linear combinations of maps $D_{a,b} : \mathbb{O} \rightarrow \mathbb{O}$, for $a, b \in \mathbb{O}$, given by

$$D_{a,b}(x) = \frac{1}{2}([a, x], b] + [a, [b, x]] + [[a, b], x]) = [[a, b], x] - 3[a, b, x]$$

where $[a, b] = ab - ba$ is the commutator and the bracket with three entries is the associator $[a, b, x] = (ab)x - a(bx)$. The associator for the octonionic algebra is totally antisymmetric.

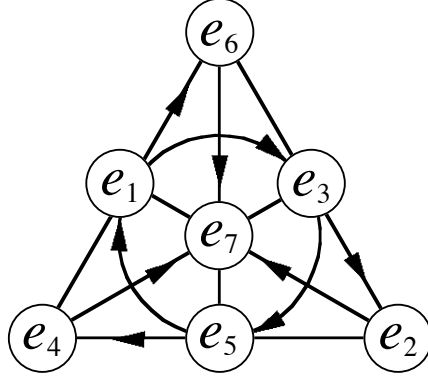


Figure 1: Fano plane representing the multiplication rule for the imaginary part of the basis for the octonions, with chosen labels.

The map satisfies the Leibniz rule

$$D_{a,b}(xy) = D_{a,b}(x)y + xD_{a,b}(y)$$

and the generalized Jacobi identity

$$D_{a,b}(D_{c,d}) = D_{D_{a,b}(c),d} + D_{c,D_{a,b}(d)} + D_{c,d}(D_{a,b}).$$

We denote the pure imaginary basis of the octonionic algebra by $e_i, i = 1, \dots, 7$ and its multiplication rule is represented by the Fano plane in Figure 1.

For each index $p \in \{1, \dots, 7\}$ consider the three pairs of indices $(i, j), (r, s)$ and (u, v) such that $e_p = e_i e_j = e_r e_s = e_u e_v$ then

$$D_{e_i, e_j} + D_{e_r, e_s} + D_{e_u, e_v} = 0 \quad (5)$$

is identically satisfied.

A basis for g_2 can be given by fourteen of these maps grouped in seven pairs, each of them constituted by D_{e_i, e_j} and D_{e_r, e_s} where $e_i e_j = e_r e_s = e_p$ for each $p \in \{1, \dots, 7\}$.

Let us now consider the transformation of the octonionic KdV equation under an infinitesimal G_2 transformation

$$u \rightarrow u + \sum_{i,j} \lambda^{ij} D_{e_i, e_j}(u) \quad (6)$$

where $\lambda^{ij} = -\lambda^{ji}$ are the infinitesimal parameters of the transformation and the summation is only over the fourteen elements of the basis of g_2 . The parameters λ^{ij} are independent of (x, t) since the G_2 symmetry is a global one.

The transformation of (2) under (6) is

$$\phi \equiv (\delta u)_t + (\delta u)_{xxx} + \frac{1}{2}(\delta u \cdot u + u \cdot \delta u)_x$$

where $\delta u = \sum_{i,j} \lambda^{ij} D_{e_i, e_j}(u)$.

Using now the Leibnitz rule for the derivation map we obtain

$$\phi = \lambda^{ij} D_{e_i, e_j} \left(u_t + u_{xxx} + \frac{1}{2}(u^2)_x \right) = 0$$

which shows that Equation (2) is invariant under the transformation given in (6).

4 Bäcklund transformation for the Cayley-Dickson KdV equations

The Cayley-Dickson (C-D) algebra of dimension 2^n contains a basis $\{e_0, e_1, \dots, e_{2^n-1}\}$ with the following relations:

$$e_0 e_0 = e_0, \quad e_0^* = e_0, \quad e_0 e_i = e_i e_0 = e_i, \quad e_i e_i = -e_0, \quad e_i^* = -e_i, \quad e_i e_j = -e_j e_i,$$

for $i, j \in \{1, \dots, 2^n - 1\}$ and $i \neq j$.

Any element x of the algebra can be expressed into its real and imaginary parts

$$x = A_0 e_0 + A_i e_i = a + \vec{A}$$

where $a^* = a$ and $\vec{A}^* = -\vec{A}$.

The associator of any three elements x, y , and z of the algebra is defined by

$$[x, y, z] = (xy)z - x(yz).$$

It is zero for associative algebras, as \mathbb{R} , \mathbb{C} , and \mathbb{H} . It is skew symmetric for alternative algebras as the octonion algebra. The C-D algebras beyond the octonions fail to have the alternative property (that is, $(x^2)y \neq x(xy)$ and $(xy)y \neq x(y^2)$, in general). However all of them are power associative algebras.

The Wahlquist-Estabrook (WE) transformation [7] for the real KdV equation can be straightforwardly generalized for the complex KdV one. We prove, in what follows, its extension for the C-D algebras. To do this we consider the KdV equation given by (2) but defined for a field $u(x, t)$ valued on a C-D algebra and we will refer this equation as the C-D KdV equation.

We define w through $u = w_x$. From (2) we obtain

$$Q(w) \equiv w_t + w_{xxx} + \frac{1}{2}w_{xx}w_x + \frac{1}{2}w_x w_{xx} = C(t)$$

where $C(t)$ is a function of t only. As in the case of the real KdV equation we can redefine

$$\tilde{w} = w - \int_{-\infty}^t C(\tau) d\tau.$$

We then obtain $Q(\tilde{w}) = 0$. In what follows we assume that this redefinition has been performed and consider $Q(w) = 0$ in order to simplify the notation.

The Bäcklund transformation for the C-D KdV equation has the expression

$$w_x + w'_x = \eta - \frac{1}{12}(w - w')^2 \quad (7)$$

$$w_t + w'_t = \frac{1}{12} \left[(w - w')^2 \right]_{xx} - \frac{1}{2} w_x^2 - \frac{1}{2} w'^2_x \quad (8)$$

where $w = w(x, t)$ and $w' = w'(x, t)$ are valued on the C-D algebra.

We now prove the following proposition.

Proposition 1 *If w and w' are solutions of (7) and (8), satisfying $\mathbb{R}e(w - w') \neq 0$, then $u = w_x$ and $u' = w'_x$ are solutions of the C-D KdV equation.*

Proof of Proposition 1 *We consider (7)_{xx} + (8), and obtain*

$$Q(w) + Q(w') = 0. \quad (9)$$

We now consider the integrability condition for (7) and (8):

$$(7)_t - (8)_x = 0.$$

We get,

$$\begin{aligned} & -\frac{1}{12} \left[(w - w')_t (w - w') + (w - w') (w - w')_t \right] - \frac{1}{12} \left[(w - w')^2 \right]_{xxx} + \\ & + \frac{1}{2} [w_{xx} w_x + w_x w_{xx}] + \frac{1}{2} [w'_{xx} w'_x + w'_x w'_{xx}] = 0. \end{aligned} \quad (10)$$

Also

$$\begin{aligned} & -\frac{1}{12} \left[(w - w')^2 \right]_{xxx} = -\frac{1}{12} (w - w')_{xxx} (w - w') - \frac{1}{4} (w - w')_{xx} (w - w')_x - \\ & - \frac{1}{4} (w - w')_x (w - w')_{xx} - \frac{1}{12} (w - w') (w - w')_{xxx}. \end{aligned} \quad (11)$$

The third and fourth terms of (10) combine with the second and third terms of the right hand side member of equation (11) to give

$$\begin{aligned} & \frac{1}{2} [w_{xx} w_x + w_x w_{xx}] + \frac{1}{2} [w'_{xx} w'_x + w'_x w'_{xx}] - \frac{1}{4} (w - w')_{xx} (w - w')_x - \\ & - \frac{1}{4} (w - w')_x (w - w')_{xx} = \frac{1}{4} (w_{xx} + w'_{xx}) (w_x + w'_x) + \frac{1}{4} (w_x + w'_x) (w_{xx} + w'_{xx}). \end{aligned} \quad (12)$$

We may now use (7) to obtain

$$\begin{aligned} & \frac{1}{4} (w_{xx} + w'_{xx}) (w_x + w'_x) + \frac{1}{4} (w_x + w'_x) (w_{xx} + w'_{xx}) = \\ & = -\frac{1}{48} [(w - w')_x (w - w') + (w - w') (w - w')_x] (w_x + w'_x) - \\ & - \frac{1}{48} (w_x + w'_x) [(w - w')_x (w - w') + (w - w') (w - w')_x]. \end{aligned} \quad (13)$$

Furthermore, using the definition of the associator,

$$\begin{aligned} (w_x + w'_x) [(w - w')_x (w - w')] &= [(w_x + w'_x) (w_x - w'_x)] (w - w') - [w_x + w'_x, w_x - w'_x, w - w'], \\ [(w - w') (w - w')_x] (w_x + w'_x) &= (w - w') [(w_x - w'_x) (w_x + w'_x)] + [w - w', w_x - w'_x, w_x + w'_x]. \end{aligned}$$

Summation of the right hand members of these two equations yields

$$\begin{aligned} & [(w_x)^2 - (w'_x)^2] (w - w') + (w - w') [(w_x)^2 - (w'_x)^2] + (w'_x w_x - w_x w'_x) (w - w') + \\ & + (w - w') (w_x w'_x - w'_x w_x) + [w - w', w_x - w'_x, w_x + w'_x] - [w_x + w'_x, w_x - w'_x, w - w']. \end{aligned}$$

In the same way

$$\begin{aligned} [(w_x - w'_x) (w - w')] (w_x + w'_x) &= (w_x - w'_x) [(w - w') (w_x + w'_x)] + [w_x - w'_x, w - w', w_x + w'_x], \\ (w_x + w'_x) [(w - w') (w_x - w'_x)] &= [(w_x + w'_x) (w - w')] (w_x - w'_x) - [w_x + w'_x, w - w', w_x - w'_x]. \end{aligned}$$

Using (7) and the power associative property of the Cayley-Dickson algebras we get

$$\begin{aligned} & (w_x - w'_x) [(w - w') (w_x + w'_x)] = (w_x - w'_x) [(w_x + w'_x) (w - w')] = [(w_x - w'_x) (w_x + w'_x)] \cdot \\ & \cdot (w - w') - [w_x - w'_x, w_x + w'_x, w - w'] = \left((w_x)^2 - (w'_x)^2 \right) (w - w') + \\ & + (w_x w'_x - w'_x w_x) (w - w') - [w_x - w'_x, w_x + w'_x, w - w'], \end{aligned}$$

$$\begin{aligned} & [(w_x + w'_x) (w - w')] (w_x - w'_x) = [(w - w') (w_x + w'_x)] (w_x - w'_x) = \\ & (w - w') \left((w_x)^2 - (w'_x)^2 \right) + (w - w') (w'_x w_x - w_x w'_x) + [w - w', w_x + w'_x, w_x - w'_x]. \end{aligned}$$

Summation of the right hand members of these two equations yields

$$\begin{aligned} & [(w_x)^2 - (w'_x)^2] (w - w') + (w - w') [(w_x)^2 - (w'_x)^2] + (w_x w'_x - w'_x w_x) (w - w') + \\ & + (w - w') (w'_x w_x - w_x w'_x) - [w_x - w'_x, w_x + w'_x, w - w'] + [w - w', w_x + w'_x, w_x - w'_x]. \end{aligned}$$

We then have from (13)

$$\begin{aligned} & \frac{1}{4} (w_{xx} + w'_{xx}) (w_x + w'_x) + \frac{1}{4} (w_x + w'_x) (w_{xx} + w'_{xx}) = -\frac{1}{24} ((w_x)^2 - (w'_x)^2) (w - w') - \\ & - \frac{1}{24} (w - w') ((w_x)^2 - (w'_x)^2) - \frac{1}{48} \{ [w - w', w_x - w'_x, w_x + w'_x] - \\ & - [w_x + w'_x, w_x - w'_x, w - w'] - [w_x - w'_x, w_x + w'_x, w - w'] + [w - w', w_x + w'_x, w_x - w'_x] + \\ & + [w_x - w'_x, w - w', w_x + w'_x] - [w_x + w'_x, w - w', w_x - w'_x] \}. \end{aligned} \quad (14)$$

We may now evaluate explicitly the associators, to do so we replace $w_x + w'_x$ by its expression given from (7), for example

$$[w_x - w'_x, w_x + w'_x, w - w'] = [w_x - w'_x, \eta, w - w'] - \frac{1}{12} [w_x - w'_x, (w - w')^2, w - w'].$$

Since η is real, the first term of the right hand side member is zero. The second term involves $w - w' = a + \vec{v}$, where a is its real part and \vec{v} its imaginary one. We have, using the properties of the basis for any C-D algebra,

$$\begin{aligned} (w - w')^2 &= a^2 + 2a\vec{v} - \|\vec{v}\|^2, \\ [w_x - w'_x, a^2 + 2a\vec{v} - \|\vec{v}\|^2, a + \vec{v}] &= [w_x - w'_x, 2a\vec{v}, \vec{v}] = 2a [w_x - w'_x, \vec{v}, \vec{v}], \end{aligned}$$

which is zero for any alternative algebra like the octonions but it is not zero for a generic C-D algebra.

However, we notice that using this result and the corresponding ones for the other associators in (14), the associators cancel by pairs: the first with the second, the third with the fifth and the fourth with the sixth.

We then obtain

$$[Q(w) - Q(w')] (w - w') + (w - w') [Q(w) - Q(w')] = 0. \quad (15)$$

If the real part of $w - w'$ is different from zero: $\mathbb{R}e(w - w') \neq 0$, the above equation implies

$$Q(w) - Q(w') = 0. \quad (16)$$

In fact, if we denote

$$\begin{aligned} w - w' &= a + \vec{v}, \\ Q(w) - Q(w') &= c + \vec{w}, \end{aligned}$$

the decomposition into its real and imaginary parts, then $(a + \vec{v})(c + \vec{w}) + (c + \vec{w})(a + \vec{v}) = 0$ imply that its real part is zero and so is its imaginary part.

That is,

$$\begin{aligned} 2ac + \vec{v}\vec{w} + \vec{w}\vec{v} &= 0, \\ 2a\vec{w} + 2c\vec{v} &= 0, \end{aligned}$$

where due to the properties of the basis of the C-D algebra $\vec{v}\vec{w} + \vec{w}\vec{v}$ is real.

If $a \neq 0$, we then get

$$\begin{aligned} \vec{w} &= -\frac{c}{a}\vec{v}, \\ 2ac + \vec{v}\vec{w} + \vec{w}\vec{v} &= \frac{2c}{a}(a^2 - \vec{v}\vec{v}) = 0 \end{aligned}$$

but $a^2 - \vec{v}\vec{v} = a^2 + \|\vec{v}\|^2 \neq 0$, hence we must have $c = 0$ and $\vec{w} = 0$, that is, equation (16).

Finally, from (9) and (16) we get

$$Q(w) = Q(w') = 0$$

and hence $u = w_x$ and $u' = w'_x$ are both solutions of the C-D KdV equation.

Remark 1: The converse of Proposition 1 is not valid. Given u and u' solutions of the KdV equation then w, w' defined through $u = w_x, u' = w'_x$ respectively, do not generally satisfy equations (7) and (8).

5 The Bäcklund transformation and the generalized Gardner equation

We will assume, as in Proposition 1, $\Re(w - w') \neq 0$. In the previous section we showed that equations (7) and (8) imply

$$Q(w) - Q(w') = 0.$$

We have

$$Q(w) - Q(w') = (w - w')_t + (w - w')_{xxx} + \frac{1}{2}(w_x)^2 - \frac{1}{2}(w'_x)^2$$

where

$$(w_x)^2 - (w'_x)^2 = \frac{1}{2}(w_x + w'_x)(w_x - w'_x) + \frac{1}{2}(w_x - w'_x)(w_x + w'_x).$$

We may now use (7) to obtain

$$(w - w')_t + (w - w')_{xxx} + \frac{1}{4} \left[\eta - \frac{1}{12}(w - w')^2 \right] (w_x - w'_x) + \frac{1}{4} (w_x - w'_x) \left[\eta - \frac{1}{12}(w - w')^2 \right] = 0. \quad (17)$$

We introduce the field $r(x, t)$ through the relation $w - w' = 2\epsilon \left(r - \frac{3}{\epsilon^2} \right)$ where $\epsilon \neq 0$ is a real parameter. We get

$$\eta - \frac{1}{12}(w - w')^2 = -\frac{1}{3}\epsilon^2 \left(r^2 - \frac{6r}{\epsilon^2} \right)$$

provided $\eta = \frac{3}{\epsilon^2}$.

We finally obtain from (17)

$$r_t + r_{xxx} + \frac{1}{2}(rr_x + r_x r) - \frac{1}{12}((r^2)r_x + r_x(r^2))\epsilon^2 = 0. \quad (18)$$

This equation, where $r(x, t)$ is valued on the C-D algebra and the product is the one of the algebra, is the generalized Gardner equation. It allows us to obtain an infinite sequence of conserved quantities, as we will show in the next section.

Proposition 2 *If w and w' are solutions of the Bäcklund equations (7) and (8) and $\mathbb{R}e(w - w') \neq 0$, then $r = \frac{3}{\epsilon^2} + \frac{1}{2\epsilon}(w - w')$ satisfy the generalized Gardner equation (18) and*

$$u = w_x = r + \epsilon r_x - \frac{1}{6}\epsilon^2 r^2 \quad (19)$$

$$u' = w'_x = r - \epsilon r_x - \frac{1}{6}\epsilon^2 r^2 \quad (20)$$

are solutions of the C-D KdV equation.

If $r(x, t)$ is a solution of the generalized Gardner equation (18), then u and u' given by (19) and (20) are solutions of the C-D KdV equation and w, w' are solutions, by choosing the integration constants in a way that $r = \frac{3}{\epsilon^2} + \frac{1}{2\epsilon}(w - w')$, of the Bäcklund equations (7) and (8).

Remark 2 We notice that u' is obtained from u by changing $\epsilon \rightarrow -\epsilon$.

Proof of Proposition 2 *From the previous argument in this section $r(x, t)$ is a solution of the Gardner equation. In addition, from (7)*

$$w_x + w'_x = 2r - \frac{1}{3}\epsilon^2 r^2, \quad (21)$$

and from the definition of r in terms of $w - w'$

$$w_x - w'_x = 2\epsilon r_x. \quad (22)$$

From these two equations we obtain (19) and (20). Proposition 1 ensures that u, u' are solutions of (2).

If $r(x, t)$ is a solution of the Gardner equation then defining w and w' from (19) and (20) we conclude that (21) and (22) are satisfied. By fixing the integration constant obtained from (22) in a way that

$$r = \frac{3}{\epsilon^2} + \frac{1}{2\epsilon}(w - w'),$$

(17) is then satisfied. Since this is the integrability condition of (7) and (8) we obtain that w and w' are solutions of (7) and (8) (we notice that (7) arises directly from (21)). We now apply Proposition 1 to show that u and u' given by (19) and (20) are solutions of (2).

6 An infinite sequence of conserved quantities

We first show that $\int_{-\infty}^{+\infty} \mathbb{R}e[r(x, t)] dx$ is a conserved quantity of the generalized Gardner equation (17).

Proposition 3 *Let $r(x, t)$ be a solution of the Gardner equation (18) and assume $r(x, t) \in \mathcal{L}(\mathbb{R})$, the Schwartz space of functions on \mathbb{R} , then $\int_{-\infty}^{+\infty} \operatorname{Re}[r(x, t)] dx$ is a conserved quantity.*

Proof of Proposition 3 *Taking the real part of equation (18), we obtain*

$$[\operatorname{Re}(r)]_t + [\operatorname{Re}(r)]_{xxx} + \frac{1}{2} [\operatorname{Re}(r^2)]_x - \frac{1}{12} \epsilon^2 \operatorname{Re}(r^2 r_x + r_x r^2) = 0.$$

We now show that $\operatorname{Re}(r^2 r_x + r_x r^2)$ is a total derivative. In fact, we consider $r = a + \vec{A}$ where a is the real part of r and \vec{A} its imaginary part. We get

$$\begin{aligned} r^2 &= a^2 + \vec{A}\vec{A} + 2a\vec{A}, \quad \vec{A}\vec{A} = -\|\vec{A}\|^2, \\ r^2 r_x + r_x r^2 &= (a^2 - \|\vec{A}\|^2 + 2a\vec{A})(a_x + \vec{A}_x) + (a_x + \vec{A}_x)(a^2 - \|\vec{A}\|^2 + 2a\vec{A}) = \\ &= 2a^2 a_x - 2\|\vec{A}\|^2 a_x + 2a(\vec{A}\vec{A}_x + \vec{A}_x \vec{A}) + 2a^2 \vec{A}_x + 2aa_x \vec{A} - 2\|\vec{A}\|^2 \vec{A}_x, \\ \operatorname{Re}(r^2 r_x + r_x r^2) &= \frac{2}{3}(a^3)_x - 2\|\vec{A}\|^2 a_x - 2a(\|\vec{A}\|^2)_x = \left(\frac{2}{3}a^3 - 2a\|\vec{A}\|^2\right)_x. \end{aligned}$$

Then

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^{+\infty} \operatorname{Re}[r(x, t)] dx &= \\ &= \int_{-\infty}^{+\infty} \left[-(\operatorname{Re}(r))_{xxx} - \frac{1}{2}(\operatorname{Re}(r^2))_x + \frac{1}{12} \epsilon^2 \left(\frac{2}{3}(\operatorname{Re}(r))^3 - 2\operatorname{Re}(r)\|\operatorname{Im}(r)\|^2 \right)_x \right] dx = 0. \end{aligned}$$

We are now able to construct, following a very well known approach [1, 11, 12, 26, 27, 28], an infinite sequence of conserved quantities. Assuming a formal expansion of r in powers of ϵ , we can invert (19). We obtain

$$\begin{aligned} r &= u - u_x \epsilon + \left(u_{xx} + \frac{1}{6} u^2 \right) \epsilon^2 - \left(u_{xx} + \frac{1}{3} u^2 \right)_x \epsilon^3 + \\ &+ \left(u_{xx} + \frac{1}{3} u^2 \right)_{xx} \epsilon^4 + \frac{1}{6} \left(uu_{xx} + \frac{1}{3} u^3 + u_{xx} u + (u_x)^2 \right) \epsilon^4 + \dots \end{aligned}$$

Using now Proposition 3 we get an infinite sequence of conserved quantities.

The first few of them are

$$\begin{aligned} H_1 &= \int_{-\infty}^{+\infty} \operatorname{Re}(u) dx, \\ H_2 &= \int_{-\infty}^{+\infty} ((\operatorname{Re}(u))^2 - \|\operatorname{Im}(u)\|^2) dx, \\ H_3 &= \int_{-\infty}^{+\infty} \left(\frac{1}{3}(\operatorname{Re}(u))^3 - (\operatorname{Re}(u_x))^2 + \|\operatorname{Im}(u_x)\|^2 - \operatorname{Re}(u)\|\operatorname{Im}(u)\|^2 \right) dx. \end{aligned}$$

Also $\int_{-\infty}^{+\infty} \mathbb{I}m(u)dx$ is a conserved quantity.

In addition to the above conserved quantities, valid for any solution $r(x, t)$ of the Gardner equation, it is also valid the following property for particular solutions of the equation.

Suppose there exists solutions of the Gardner equation for which $\|\mathbb{I}m(r)\|^2$ is constant. That is, if $r = a + \vec{A}$, suppose there are solutions such that $\vec{A}\vec{A}_x + \vec{A}_x\vec{A} = 0$. Then $\int_{-\infty}^{+\infty} \mathbb{I}m(r)dx$ is conserved. In fact, for these particular solutions we have

$$(r^2) r_x + r_x (r^2) = (r^3)_x - \left(2\|\mathbb{I}m(r)\|^2 \vec{A} \right)_x,$$

where $r^3 = a^3 - 3a\|\mathbb{I}m(r)\|^2 + 3a^2\vec{A} - \|\mathbb{I}m(r)\|^2\vec{A}$ and $\|\mathbb{I}m(r)\|^2 = -\vec{A}\vec{A}$.

7 Conclusions

We introduced a Korteweg-de Vries equation valued on a Cayley-Dickson algebra and obtained for it an infinite sequence of local conserved quantities. To do this we defined a Bäcklund transformation in the sense of Walhquist-Estabrook and obtained the corresponding generalized Gardner equation. Then the conserved quantities follow directly by inverting the Gardner transformation. All of these results hold in particular for the quaternion, octonion and sedenion Korteweg-de Vries equations which have additional symmetries associated to the automorphisms of the corresponding algebras, for example the octonion KdV equation is invariant under the Lie group G_2 . A property which is crucial in the proof is that Cayley-Dickson algebras obey the power associative property, that is, the associativity for every power of arbitrary element in the algebra.

Due to the connection of the octonion algebra to supersymmetric theories an interesting problem to consider would be the supersymmetric extension of the octonion KdV equation and its relation to the octonionic truncation of the $D = 11$ Supermembrane.

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